On weak determinacy of infinite binary games(Study of definability in nonstandard models of arithmetic)

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On weak determinacy of infinite binary games

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概要

In [5] and [6], we have investigated the logical strength of the determinacy of infinite games in the Baire space up to Δ_3^0. In this paper we consider infinite games in the Cantor space. Let Det^* (resp. Det) stand for the determinacy of infinite games in the Cantor space (resp. the Baire space). In Section 2, we show that Δ_1^0-Det^*, Σ_1^0-Det^* and WKL_0 are pairwise equivalent over RCA_0. In Section 3, we show that $\mathsf{RCA}_0 + (\Sigma_1^0 \wedge \Pi_1^0)$-$\mathsf{Det}^*$ is equivalent to ACA_0. Then, we deduce that $\mathsf{RCA}_0 + \Delta_2^0$-$\mathsf{Det}^*$ is equivalent to ATR_0. In the last section, we show some more equivalences among stronger assertions without details, which will be thoroughly treated elsewhere.

1 Preliminaries

In this section, we recall some basic definitions and facts about second order arithmetic. The language $\mathcal{L}_2$ of second order arithmetic is a two-sorted language with number variables $x, y, z, \ldots$ and unary function variables $f, g, h, \ldots$, consisting of constant symbols $0, 1, +, \cdot, =, <$. We also use set variables $X, Y, Z, \ldots$, intending to range over the set of $\{0,1\}$-valued functions, that is, characteristic functions of sets.

The formulae can be classified as follows:

- φ is *bounded* (Π_0^0) if it is built up from atomic formulae by using propositional connectives and bounded number quantifiers $(\forall x < t), (\exists x < t)$, where t does not contain x.

- φ is Π^1_0 if it does not contain any function quantifier. Π^1_0 formulae are called *arithmetical* formulae.
- $\neg\varphi$ is Σ^i_n if φ is a Π^i_n formula ($i \in \{0,1\}, n \in \omega$).
- $\forall x_1 \cdots \forall x_k \varphi$ is Π^0_{n+1} if φ is a Σ^0_n formula ($n \in \omega$),
- $\forall f_1 \cdots \forall f_k \varphi$ is Π^1_{n+1} if φ is a Σ^1_n formula ($n \in \omega$).

Using above classification, we can define schemata of comprehension and induction as follows.

Definition 1.1 Assume $n \in \omega$ and $i \in \{0,1\}$. The scheme of Π^i_n *comprehension*, denoted Π^i_n-CA, consists of all the formulae of the form

$$\exists X \forall x (x \in X \leftrightarrow \varphi(x)),$$

where $\varphi(x)$ belongs to Π^i_n and X does not occur freely in $\varphi(x)$. The scheme of Δ^i_n*-comprehension*, denoted Δ^i_n-CA, consists of all the formulae of the form

$$\forall n(\varphi(n) \leftrightarrow \psi(n)) \;\rightarrow\; \exists X \forall n(n \in X \leftrightarrow \varphi(n))$$

where $\varphi(n)$ is Σ^i_n, $\psi(n)$ is Π^i_n, and X is not free in $\varphi(n)$. The scheme of Σ^i_n *induction*, denote Σ^i_n-IND, consists of all axioms of the form

$$(\varphi(0) \wedge \forall n(\varphi(n) \;\rightarrow\; \varphi(n+1))) \;\rightarrow\; \forall n \phi(n)$$

where $\varphi(n)$ is Σ^i_n.

Now we define a basic subsystem of second order arithmetic, called RCA_0.

Definition 1.2 RCA_0 is the formal system in the language of $\mathcal{L}_2$ which consists of discrete order semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemata of Δ^0_1 comprehension and Σ^0_1 induction.

The following is a formal version of the *normal form theorem* for Σ^0_1 relations.

Theorem 1.3 (normal form theorem) *Let $\varphi(f)$ be a Σ^0_1 formula. Then we can find a Π^0_0 formula $R(s)$ such that* RCA_0 *proves*

$$\forall f(\varphi(f) \;\leftrightarrow\; \exists m R(f[m]))$$

where $f[m]$ is the code for the finite initial segment of f with length m. Note that $\varphi(f)$ may contain free variables other than f.

Proof. See also Simpson [7, Theorem II.2.7]. □

We loosely say that a formula is Σ^i_n (resp. Π^i_n) if it is equivalent over a base theory (such as RCA_0) to a $\psi \in \Sigma^i_n$ (resp. Π^i_n).

The next theorem asserts that the universe of functions is closed under the *least number operator*, i.e., *minimization.*

Theorem 1.4 (minimization) *The following is provable in* RCA_0. *Let* $f : \mathbb{N}^{k+1} \to \mathbb{N}$ *be such that for all* $\langle n_1, ... n_k \rangle \in \mathbb{N}$ *there exists* $m \in \mathbb{N}$ *such that* $f(m, n_1, ... n_k) = 1$. *Then there exists* $g : \mathbb{N}^k \to \mathbb{N}$ *such that* $g(n_1, ... n_k)$ *is the least* m *such that* $f(m, n_1, ..., n_k) = 1$.

Proof. See Simpson [7, Theorem II.3.5]. □

2 WKL_0 and Σ^0_1-Det*

Let X be either $\{0, 1\}$ or $\mathbb{N}$ and let φ be a formula with a distinct variable f ranging over $X^{\mathbb{N}}$. A two-person *game* G_φ (or simply φ) over $X^{\mathbb{N}}$ is defined as follows: player I and player II alternately choose elements from X (starting with I) to form an infinite sequence $f \in X^{\mathbb{N}}$ and I (resp. II) wins iff $\varphi(f)$ (resp. $\neg\varphi(f)$). A *strategy* of player I (resp. II) is a map $\sigma : X^{\text{even}} = \{s \in X^{<\mathbb{N}} | s \text{ has even length}\} \to X$ (resp. $X^{\text{odd}} \to X$). We say that φ is *determinate* if one of the players has a *winning strategy*, that is, a strategy σ such that the player is guaranteed to win every play f in which he played $f(n) = \sigma([f(n)])$ whenever it was his turn to play.

Given a class of formulae $\mathcal{C}$, $\mathcal{C}$-determinacy is the axiom scheme which states that any game in $\mathcal{C}$ is determinate. We use $\mathcal{C}$-Det* (resp $\mathcal{C}$-Det) to denote $\mathcal{C}$-determinacy in the Cantor space (resp. the Baire space).

A set T of finite sequences is a *tree* if it is closed under initial segment, i.e., $t \in T$ and $s \subseteq t$ implies $s \in T$. A function f is a *path* of T if each initial segment of f is a sequence of T.

Definition 2.1 WKL_0 is a subsystem of second order arithmetic whose axioms are those of RCA_0 plus weak König's lemma which states that every infinite binary tree $T \subseteq 2^{<\mathbb{N}}$ has an infinite path.

Next, we prove the equivalences among WKL_0, Σ^0_1-Det* and Δ^0_1-Det*.

Theorem 2.2 $\mathsf{RCA}_0 \vdash \Delta^0_1\text{-Det}^* \to \mathsf{WKL}_0$.

Proof. By way of contradiction, we assume $\mathsf{RCA}_0 + \Delta^0_1$-$\mathsf{Det}^*$ and deny weak König's lemma. Let T be an infinite binary tree in which there is no infinite path, i.e., there is no f such that $\forall n f[n] \in T$. We consider the following game:

- Player I plays a sequence t of $2^{<\mathbb{N}}$.
- Player II then answers by playing 0 or 1.
- Player I plays a new sequence u of $2^{<\mathbb{N}}$.
- Player II then plays a sequence v of $2^{<\mathbb{N}}$.

The winning conditions of the game are given as follows: II wins if one of the following cases holds.

- $t * \langle i \rangle * u \notin T$.
- $t * \langle 1-i \rangle * v \in T \wedge |u| \leq |v|$.

We shall remark that the game always terminates in finite moves, because T has no infinite path. This ensures that the game is Δ^0_1. On the other hand, we can show that player I has no winning strategy by considering two cases, in one of which player II chooses $i = 0$ and in the other he chooses $i = 1$ after player I plays t. I can not win in both of them. Therefore, by Δ^0_1-Det^* player II has a winning strategy τ. Using τ, we define $f : \mathbb{N} \to \{0, 1\}$ as follows:

- $f(0) = 1 - \tau(\langle\rangle)$,
- $f(n+1) = 1 - \tau(f[n])$,

By Σ^0_0-induction, we can easily see that $f[n] \in T$ for all n, which contradicts with our assumption that T has no infinite path. Thus, Δ^0_1-$\mathsf{Det}^* \to \mathsf{WKL}$. This completes the proof of the theorem.□

Now, we turn to prove the reversal.

Theorem 2.3 $\mathsf{WKL}_0 \vdash \Sigma^0_1$-$\mathsf{Det}^*$.

Proof. Let $\varphi(f)$ be a Σ^0_1-formula with $f \in 2^{\mathbb{N}}$. Then, by the normal form theorem, $\varphi(f)$ can be written as $\exists n R(f[n])$, where R is Π^0_0. We define recursive maps g and g_n from $2^{<\mathbb{N}}$ to $\{0, 1\}$ for each $n \in \mathbb{N}$ as follows:

$$g(s) = \begin{cases} 1 & \text{if } \exists t \subseteq s\, R(t) \\ 0 & \text{if } \forall t \subseteq s\, \neg R(t) \end{cases}$$

$$g_n(s) = \begin{cases} g(s) & \text{if } |s| \geq n \\ \max\{g_n(s * \langle 0 \rangle), g_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\ \min\{g_n(s * \langle 0 \rangle), g_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} \end{cases}$$

Intuitively, for $n \in \mathbb{N}$, $g_n(\langle\rangle) = 1$ means "player I can win the game by stage n," and $g_n(\langle\rangle) = 0$ means "player I cannot win by stage n."

Claim. The following assertions hold.

(1) If $g_n(\langle\rangle) = 1$ for some n, then I has a winning strategy.

(2) If $g_n(\langle\rangle) = 0$ for every n, then II has a winning strategy.

For (1), fix n such that $g_n(\langle\rangle) = 1$. Define $\sigma : 2^{\text{even}} \to \{0, 1\}$ by

$$\sigma(s) = \begin{cases} 0 & \text{if } g_n(s * \langle 0 \rangle) = 1 \\ 1 & \text{otherwise.} \end{cases}$$

We can verify that σ is a winning strategy for player I, which completes the proof of the first assertion of the claim.

For (2), suppose that for any n, $g_n(\langle\rangle) = 0$ and show that player II has a winning strategy. The idea of the proof is as follows. Consider an infinite binary tree which consists of the moves at which player II can prevent player I from winning the game. A path through such a tree will serve a winning strategy for II. To realize this idea, we will need some coding arguments to construct the tree.

To begin with, fix a lexicographical enumeration e of non-empty sequences of $2^{<\mathbb{N}}$. For instance, $e(\langle 0 \rangle)$=0, $e(\langle 1 \rangle) = 1$, $e(\langle 0, 0 \rangle) = 2$, and so on. Using e, we can regard any $s \in 2^{<\mathbb{N}}$ as a partial strategy (i.e., a finite segment of the strategy) for player II (cf. [1]). We define T_s to be the tree consisting of all partial plays in which player II follows s. More precisely, T_s is defined as follows:

$$t \in T_s \leftrightarrow \forall k(2k + 1 < |t| \to t(2k + 1) = s(e(\langle t(0), \cdots, t(2k) \rangle))).$$

Finally we define T, a set of all moves which avoid the winning of player I, as follows:

$$s \in T \leftrightarrow \forall t \in T_s g_{h(s)}(t) = 0,$$

where $h : 2^{<\mathbb{N}} \to \mathbb{N}$ is defined by $h(s) = \max\{|t| : t \in T_s\}$. Clearly T is recursive, therefore it exists in RCA_0. On the other hand, the assumption $\forall n\, g_n(\langle\rangle) = 0$ implies that T is infinite. Thus, T has a infinite path f by weak König's lemma.

Now, we define $\tau : 2^{\text{odd}} \to \mathbb{N}$ as:

$$\tau(s) = f(e(\langle s(0)...s(|s| - 2) \rangle)),$$

and then we can verify that τ is a winning strategy for player II, which completes the proof. □

3 ATR_0 and Δ^0_2-Det*

In this section we aim to show that $\mathsf{RCA}_0 + \Delta^0_2$-Det* and ATR_0 are equivalent. We first give the definitions of ACA_0 and ATR_0.

Definition 3.1 The system ACA_0 consists of the discrete order semi-ring axioms for $(\mathbb{N}, +, \cdot, 0, 1, <)$ plus the schemes of Σ^0_1 induction and arithmetical comprehension.

Since comprehension axioms admit free variables, Π^0_1 comprehension is already as strong as arithmetical comprehension.

Lemma 3.2 *The following are pairwise equivalent over* RCA_0.
(1) arithmetical comprehension;
(2) Π^0_1 *comprehension.*

Proof. See Simpson [7, Lemma III.1.3]. □

Definition 3.3 ATR_0 consists of RCA_0 augmented by the following axiom, called *arithmeticat transfinite recursion:* For any set $X \subset \mathbb{N}$ and for any well-ordering relation $\prec$, there exists a set $H \subset \mathbb{N}$ such that

- if b is the $\prec$-least element, then $(H)_b = X$,
- if b is the immediate successor of a w.r.t. $\prec$, then $\forall n(n \in (H)_b \leftrightarrow \psi(n, (H)_a))$,
- if b is a limit, then $\forall a \forall n((n, a) \in (H)_b \leftrightarrow (a \prec b \wedge n \in (H)_a))$,

where ψ is a Π^1_0-formula and $(H)_a = \{x : (x, a) \in H\}$, where (x, b) denotes the code of the pair $\langle x, a\rangle$.

ATR_0 is obviously stronger than ACA_0, but it is contained in Π^1_1-CA_0.

Lemma 3.4 *The following are pairwise equivalent over* RCA_0*:*

$$\Delta^0_1\text{-Det},\ \Sigma^0_1\text{-Det } \textit{and } \mathsf{ATR}_0.$$

Proof. See [7] or [8].

The class $\Sigma^0_1 \wedge \Pi^0_1$ is defined as follows. φ is $\Sigma^0_1 \wedge \Pi^0_1$ if and only if φ is of the form $\psi_0 \wedge \neg\psi_1$, where ψ_0 and ψ_1 are Σ^0_1. The following theorems characterize $(\Sigma^0_1 \wedge \Pi^0_1)$ determinacy in the Cantor space.

Theorem 3.5 ACA_0 *proves* $(\Sigma^0_1 \wedge \Pi^0_1)$-Det*.

Proof. Let φ be of the form $\exists n R_0(f[n]) \wedge \forall n R_1(f[n])$. We define the functions g, g_n, g', and g'_n from $2^{<\mathbb{N}}$ to $\{0, 1\}$ as follows:

- $g(s) = \begin{cases} 1 & \text{if } \exists t \subseteq s\, R_0(t) \\ 0 & \text{if } \forall t \subseteq s\, \neg R_0(t) \end{cases}$

- $g_n(s) = \begin{cases} g(s) & \text{if } |s| \geq n \\ \max\{g_n(s * \langle 0 \rangle), g_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\ \min\{g_n(s * \langle 0 \rangle), g_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} \end{cases}$

- $g'(s) = \begin{cases} 1 & \text{if } \forall t \subseteq s\, R_1(t)) \\ 0 & \text{if } \exists t \subseteq s\, \neg R_1(t)) \end{cases}$

- $g'_n(s) = \begin{cases} g'(s) & \text{if } |s| \geq n \\ \max\{g'_n(s * \langle 0 \rangle), g'_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is even} \\ \min\{g'_n(s * \langle 0 \rangle), g'_n(s * \langle 1 \rangle)\} & \text{if } |s| < n \text{ and } |s| \text{ is odd} \end{cases}$

Following a similar argument of the one used in the proof of Theorem 2.2, we can prove

Claim: if there exists n such that $g_n(\langle\rangle) \cdot g'_m(\langle\rangle) = 1$ for all $m > n$ then I has a winning strategy, otherwise player II has a winning strategy.

This complete the proof of the theorem. □

Theorem 3.6 $\mathsf{RCA}_0 \vdash (\Sigma^0_1 \wedge \Pi^0_1)\text{-}\mathsf{Det}^* \rightarrow \mathsf{ACA}_0$

Proof. Let $\varphi(n)$ be a Σ^0_1-formula. We need to construct a set X such that for any $n \in \mathbb{N}$, $\varphi(n) \leftrightarrow n \in X$. To construct X, consider the following game: player I asks II about n by playing 0 consecutively n times and playing 1 after that (if he plays 0 for ever, he loses). II ends the game by answering 0 or 1.

Now, suppose that player I plays n $0'$s and a 1 consecutively. Player II wins if one of the following cases holds.

- II answers 1 and $\varphi(n)$.
- II answers 0 and $\neg\varphi(n)$.

Clearly, I has no wining strategy. By $(\Sigma^0_1 \wedge \Pi^0_1)$-Det*, let τ be a winning strategy of player II. We defined a set X by:

$$n \in X \leftrightarrow \tau(0^n 1) = 1.$$

The set X exists by Π^0_0 comprehension. Moreover, we can verify that $\forall n$, $\varphi(n) \leftrightarrow n \in X$, which completes the proof. □

Let $\prec$ be a recursive well-ordering on $\mathbb{N}$. We define a recursive well-ordering $\prec^*$ on $\mathbb{N} \times \{0, 1\}$ as follows:

$$(x, i) \prec^* (y, j) \text{ iff } x \prec y \vee (x = y \wedge i < j).$$

Let X be either $\mathbb{N}$ or $\{0,1\}$. We say that a formula $\varphi(n,i,f)$ with distinct free variable f ranging over $X^{\mathbb{N}}$ is *decreasing along* $\prec^*$ if and only if

$$\forall n \forall i \forall m \forall j\,(((m,j) \prec^* (n,i) \wedge \varphi(n,i,f)) \rightarrow \varphi(m,j,f)),$$

for all f.

The following lemma will play a key role to characterize Δ^0_2-Det^*.

Lemma 3.7 *It is provable in* RCA_0 *that a formula* ψ *is* Δ^0_2 *if and only if:*

$$\psi(f) \leftrightarrow \exists x(\varphi(x,0,f) \wedge \neg\varphi(x,1,f)),$$

where φ *is* Π^0_1 *and it is decreasing along some recursive well-ordering relation* $\prec^*$.

Proof. See [8] for the proof. □

Theorem 3.8 ATR_0 *is equivalent to* $\mathsf{RCA}_0 + \Delta^0_2$-$\mathsf{Det}^*$.

Proof. The proof is a modification of the proof of Theorem 6.1 in [8]. By Theorem 3.6 and Lemma 3.7, Δ^0_2-Det^* is just a transfinite iteration of arithmetical comprehension, which is the same as ATR_0. □

4 Further classes of games

In this section, we summarize our results about the determinacy of Boolean combinations of Σ^0_2-games. The detailed treatment of these results will appear in our forthcoming paper.

We start by formalizing the inductive definition of a class of operators.

Definition 4.1 *Given a a class of formulas* $\mathcal{C}$*, the axiom* $\mathcal{C}$-ID *asserts that for any operator* $\Gamma \in \mathcal{C}$*, there exists* $W \subset \mathbb{N} \times \mathbb{N}$ *such that*

1. *W is a pre-wellordering on its field F,*
2. $\forall x \in F \quad W_x = \Gamma(W_{<x}) \cup W_{<x}$,
3. $\Gamma(F) \subset F$.

For a class of formulas $\mathcal{C}$, Γ is a monotone $\mathcal{C}$-operator if and only if $\Gamma \in \mathcal{C}$ and Γ satisfies $\Gamma(X) \subset \Gamma(Y)$ whenever $X \subset Y$. The class of monotone $\mathcal{C}$-operators is denoted by mon-$\mathcal{C}$. We also use $\mathcal{C}$-MI to denote [mon-$\mathcal{C}$]-ID. We refer the reader to our papers [9], [5] for more information on this formalization.

Theorem 4.2 *The following assertions hold over* RCA_0.

(1) Σ^0_2-$\mathsf{MI} \rightarrow \Sigma^0_2$-$\mathsf{Det}^*$.

(2) Σ_2^0-Det* $\to \Sigma_2^0$-ID.

Proof. The idea of the proof is similar to the one used in [9] and [5]. We just mention that since the game is played over the Cantor space, rather than the Baire space, we can replace the Σ_1^1-operator in [9] and [5] by a Σ_2^0-operator.□

Now, we turn to investigate the strength of Σ_2^0-ID. The following lemma provides an alternative definition of Π_1^1-CA$_0$.

Lemma 4.3 *The following assertions hold over* RCA$_0$.

(1) Π_1^1-CA $\leftrightarrow (\Sigma_1^0 \wedge \Pi_1^0)$-Det.

(2) Π_1^0-MI $\to \Pi_1^1$-CA.

Proof. The proof of the assertion (1) can be found either in [8] or in [7]. The assertion (2) is a straightforward formalization of Hinman's proof [4]. □

Theorem 4.4 Π_1^1-CA $\vdash \Pi_1^1$-MI.

Proof. Let Γ be a monotone Π_1^1-operator. Using the strategy of a certain $(\Sigma_1^0 \wedge \Pi_1^0)$-game, we can construct W which satisfys conditions (1), (2) and (3) of Definition 4.1. This completes the proof by the assertion (1) of Lemma 4.3.□

Finally, we give the following corollary.

Corollary 4.5 *The following are equivalent over* RCA$_0$*:*

Σ_2^0-Det*, Π_1^1-CA$_0$, Π_1^0-MI, Σ_2^0-ID *and* Π_1^1-MI.

Proof. It is straightforward from Theorems 4.2 and 4.4.□

Next, we turn to the games which can be written as Boolean combinations of Σ_2^0-formulas. We first recall the following definitions from [6]. The class $(\Sigma_2^0)_k$ of formulas is defined as follows. For $k = 1$, $(\Sigma_2^0)_1 = \Sigma_2^0$. For $k > 1, \psi \in (\Sigma_2^0)_k$ iff it can be written as $\psi_1 \wedge \psi_2$, where $\neg\psi_1 \in (\Sigma_2^0)_{k-1}$ and $\psi_2 \in \Sigma_2^0$. It can be shown that for any formula ψ in the class of Boolean combinations of Σ_2^0-formulas, there is a $k \in \omega$ such that $\psi \in (\Sigma_2^0)_k$, or more strictly, ψ is equivalent to a formula in $(\Sigma_2^0)_k$.

Theorem 4.6 *Assume* $0 < k < \omega$. *Then,* $(\Sigma_2^0)_{k+1}$-Det* $\leftrightarrow (\Sigma_2^0)_k$-Det.

Proof. $(\to)$. Let ψ be a $(\Sigma_2^0)_k$-formula and G_ψ the infinite game over $\mathbb{N}^\mathbb{N}$ associated with ψ. We explain how to turn G_ψ to a $(\Sigma_2^0)_{k+1}$-game over $2^\mathbb{N}$, which will be denoted G_ψ^*. The idea is the following: In G_ψ^*, I starts by playing n_0 0's, then plays 1. Then, II plays n_1 1's and plays 0 and so on. We need to avoid some trivial situation. For instance, player I must not play 0's consecutively for ever. He must stop after playing finitely may 0's to give II a chance to play. This will make G^*_ψ a $(\Sigma^0_2)_{k+1}$-game and hence determinate by our assumption. On the other hand the player who wins G^*_ψ can win G_ψ, which completes the proof of the first direction.

The direction $(\leftarrow)$ can be proved by using the inductive definition of a combination of k Σ^1_1-operators, which is equivalent to $(\Sigma^0_2)_k$-Det by [6]. □

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